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Abstract. It is shown that in a rotational coordinate system the elementary solutions of Maxwell's equations can be derived from two scalar functions.

These two functions are the azimuthal (angular) components E_ϕ and H_ϕ of an electromagnetic field and satisfy a differential equation of fourth order, which in particular cases can be factorized.

This result follows from a theorem which states that in a wide class of rotational coordinates the azimuthal component F_ϕ satisfies the following differential equation:

$$\left(2^{-2}h_3^4(\Delta+k^2-h_3^{-2})^2+\frac{\partial^2}{\partial\phi^2}\right)F_\phi=0,$$

if the field $\boldsymbol{F}$ satisfies the vector Helmholtz equation.

1. Introduction

It is known that the theory of the vector Helmholtz (VH) equation in a curvilinear orthogonal coordinate system is more complicated than the theory of the scalar Helmholtz (SH) equation since the breaking of a vector equation into scalar equations, each involving only one unknown curvilinear component leads to higher-order differential equations which are difficult to handle (Morse and Feshbach 1953, p 1761).

Therefore in electromagnetic theory, the most effective procedure, useful in principle in each coordinate system, starts from the Cartesian components of the electromagnetic field, which satisfy the SH equation.

Other more elegant methods introduce potentials, particularly the Hertz potential, and sometimes they allow decomposition of the initial vector problem into independent scalar equations.

Unfortunately, such a scalarization cannot always be effected even in those coordinate systems in which the SH equation is separable (Morse and Feshbach 1953, Moon and Spencer 1961, Przeździecki 1960).

However, there exists a wide class of orthogonal rotational coordinate systems in which the azimuthal components of the fields play an important role.

In the present paper, we will show that in any system of this class:

(i) if the vector field $\boldsymbol{F}$ satisfies the homogeneous VH equation, the azimuthal (angular) component F_ϕ satisfies a fourth-order differential equation;

(ii) if, furthermore, this azimuthal component F_ϕ is in the form

$$F_\phi=\sum_m F_\phi^m=\sum_m f_m(\xi_1,\xi_2)\,\mathrm{e}^{\mathrm{i}m\phi},\tag{1}$$

then the coefficients f_m obviously also satisfy a fourth-order equation, which can now be factorized into two second-order equations which bear close resemblance to the SH equation.

(iii) If the fields $\boldsymbol{E}^m$ and $\boldsymbol{H}^m$ also satisfy the Maxwell equations and the azimuthal components E_Φ^m, H_Φ^m are in the form $f_m\, e^{im\phi}$, $h_m\, e^{im\phi}$, respectively, then the other components of the fields $\boldsymbol{E}^m$, $\boldsymbol{H}^m$ can be derived from E_Φ^m, H_Φ^m (although, as a rule each field component is determined by both E_Φ^m and H_Φ^m).

Thus, the vector problem can be reduced to the scalar one. The properties of the coordinate systems in which (i)–(iii) hold will be specified in § 2.

2. General geometrical properties

We shall consider a three-dimensional, curvilinear, orthogonal coordinate system with the coordinates ξ_1, ξ_2, ξ_3, related to the Cartesian frame by the equations

$$x = r(\xi_1, \xi_2)\cos\xi_3,$$
$$y = r(\xi_1, \xi_2)\sin\xi_3, \tag{2}$$
$$z = z(\xi_1, \xi_2),$$

i.e. it is obtained by the rotation of the two-dimensional orthogonal system ξ_1, ξ_2 about the z axis. The coordinate ξ_3 is identified with the rotation angle ϕ and sometimes we will write $\xi_3 = \phi$.

We restrict our considerations only to such plane systems which represent conformal mappings $\xi_1 + i\xi_2 = g(r + iz)$, where r and z are radial and longitudinal coordinates, respectively, in the cylindrical coordinate system.

Fixing the z axis, let us denote by R the class of the rotational coordinate systems ξ_1, ξ_2, ξ_3 related to the Cartesian frame by (2), if ξ_1 and ξ_2 are real and imaginary parts of some analytic function $g(r + iz)$.

The circular cylinder, spherical, parabolic, prolate and oblate spheroidal, etc, coordinates belong to the class R, if independent variables ξ_1, ξ_2 are suitably chosen. This choice concerns only a proper scale of each coordinate (Morse and Feshbach 1953, p 504). For instance in the spherical coordinates R, Θ, Φ it is sufficient to substitute $\xi_1 = \ln R$, $\xi_2 = \frac{1}{2}\pi - \Theta$, as in this case $r + iz = R\sin\Theta + iR\cos\Theta = \exp[\ln R + i(\frac{1}{2}\pi - \Theta)] = \exp(\xi_1 + i\xi_2)$.

The coordinate systems which belong to R have some special properties, which are fundamental for our considerations.

First of all, the scale factors defined as (Morse and Feshbach 1953, pp 24, 115)

$$h_i = g_{ii} = [(x_{,i})^2 + (y_{,i})^2 + (z_{,i})^2]^{1/2} \tag{3}$$

(where $(\ldots)_{,i} = \partial/\partial\xi_i$, $i = 1, 2, 3$) are

$$h_i = [(r_{,i})^2 + (z_{,i})^2]^{1/2}, \qquad i = 1, 2; \qquad h_3 = r. \tag{4}$$

Moreover

$$(r_{,1})^2 = (z_{,2})^2; \qquad (r_{,2})^2 = (z_{,1})^2, \tag{5}$$

from analyticity of the conformal mapping $\xi_1 + i\xi_2 = g(r + iz)$, and

$$h_1 = h_2 = [(r_{,1})^2 + (r_{,2})^2]^{1/2} = h. \tag{6}$$

It is also easy to check that

$$\Delta h_3 = h_3^{-1}; \qquad (\text{grad } h_3)^2 = 1. \tag{7}$$

3. The fourth-order equation

To prove theorem (i), observe that in each system of the class R, the relation between the azimuthal and Cartesian components of a vector $\boldsymbol{F}$ is

$$F_\phi = -F_x \sin\phi + F_y \cos\phi. \tag{8}$$

It is convenient also to employ an auxiliary function

$$F_r = F_x \cos\phi + F_y \sin\phi, \tag{9}$$

which in fact is the radial component of the same vector $\boldsymbol{F}$ in circular-cylinder coordinates. Applying the operator $\Delta = \text{div grad}$ to (8), we get

$$\begin{aligned}\Delta F_\phi &= -[\sin\phi\, \Delta F_x + 2(\text{grad } F_x)(\text{grad} \sin\phi) + F_x\, \Delta \sin\phi] \\ &\quad + [\cos\phi\, \Delta F_y + 2(\text{grad } F_y)(\text{grad} \cos\phi) + F_y\, \Delta \cos\phi] \\ &= -(k^2 + h_3^{-2})F_\phi - 2h_3^{-2}(F_{x,\phi} \cos\phi + F_{y,\phi} \sin\phi),\end{aligned} \tag{10}$$

if $\boldsymbol{F}$ satisfies the VH equation. But, differentiating (8) and (9) we obtain

$$F_{\phi,\phi} = -F_{x,\phi} \sin\phi + F_{y,\phi} \cos\phi - F_r, \tag{11}$$

$$F_{r,\phi} = F_{x,\phi} \cos\phi + F_{y,\phi} \sin\phi + F_\phi, \tag{12}$$

therefore (10) becomes

$$Q(F_\phi) = -F_{r,\phi}, \tag{13}$$

where Q denotes the differential operator

$$Q = 2^{-1} h_3^2 (\Delta + k^2 - h_3^{-2}). \tag{14}$$

Similarly (9), (12) and (8) yield

$$Q(F_r) = F_{\phi,\phi}. \tag{15}$$

Now, since none of the h_i coefficients depend on ϕ, the operators Q and $(\ldots)_{,\phi}$ commute. Therefore, one can eliminate the auxiliary function F_r to find

$$\left(Q^2 + \frac{\partial^2}{\partial\phi^2}\right) F_\phi = 0. \tag{16}$$

In conclusion, if the field $\boldsymbol{F}$ satisfies the VH equation, the azimuthal component F_ϕ in any coordinate system of the class R satisfies the fourth-order partial differential equation (16). It is seen that F_r also satisfies this equation.

Since from general theory it follows only that field components satisfy a differential equation of an order not higher than six (Morse and Feshbach 1953, p 1761), the above result can be of some advantage.

Further simplifications occur in some special cases. For example, if F_ϕ is given by (1), then substituting (1) into (16), we find that

$$L_{m-1} L_{m+1} f_m = L_{m+1} L_{m-1} f_m = 0, \tag{17}$$

where the operator L_q is defined as

$$L_q = 2^{-1}h_3^2(\Delta_2 + k^2 - q^2h_3^{-2}), \tag{18}$$

or

$$L_q(f) = e^{-iq\phi}(Q+\tfrac{1}{2})(f\, e^{iq\phi}) \qquad q = 0, \pm 1, \pm 2, \ldots,$$

and Δ_2 represents the first two terms of the Laplace operator in R coordinates,

$$\Delta_2 = \frac{1}{h^2h_3}\{[h_3(\ldots)_{,1}]_{,1} + [h_3(\ldots)_{,2}]_{,2}\}. \tag{19}$$

Observe from (18) that

$$L_q = L_{-q}; \qquad L_{q-1} = L_{q+1} + 2q; \qquad q = 0, \pm 1, \pm 2, \ldots. \tag{20}$$

Let us now examine the solutions of (17). To this purpose let us denote by $g_m^\pm$ the solutions of the equation

$$L_{m\pm 1}g_m^\pm = 0, \qquad \text{with } m \text{ an integer.} \tag{21}$$

For fixed m, the second-order differential operators $L_{m\pm 1}$ can have discrete and/or continuous spectra $S_{m\pm 1}$.

If $g_m^\pm(s)$, $s \in S_{m+1}$, constitute the base in the space $T_{m\pm 1}$ of all solutions of (21), each solution can be written as

$$f_m^\pm = \int_{S_{m\pm 1}} \gamma_m^\pm(s)g_m^\pm(s)\, ds, \tag{22}$$

where the integration is over the spectrum $S_{m\pm 1}$.

Obviously, the integration is reduced to a sum in the case of a discrete spectrum in which case we substitute

$$\gamma_m^\pm(s) = \sum_n \gamma_{mn}^\pm \delta(s-n), \tag{23}$$

where the last term denotes the Dirac δ function.

It is evident from (20) that we can choose a set of solutions such that

$$g_{-m}^\pm = g_m^\pm \qquad \text{and} \qquad g_m^- = g_{m-2}^+. \tag{24}$$

Therefore the set of the independent solutions may be written, for example, as

$$g_0^+, g_0^-, g_1^+, g_1^-, g_2^+, g_3^+, \ldots. \tag{25}$$

Thus equation (17) is satisfied either by the function $f_m^\pm$ in the form (22), where $g_m^\pm \in T_{m\pm 1}$, or by the functions $\hat{g}_m$, which are non-trivial solutions of the non-homogeneous equation

$$L_{m-1}\hat{g}_m = \int \gamma_m(s)g_m^+(s)\, ds, \tag{26}$$

where $\gamma_m(s)$ an arbitrary function, and $g_m^+ \in T_{m+1}$.

However, with the aid of (20) $(m \neq 0)$, we see that just g_m^+ satisfies the non-homogeneous equation

$$L_{m-1}g_m^+ = 2mg_m^+. \tag{27}$$

From the comparison of (26) and (27) via (21), it appears that the functions $\hat{g}_m$ do not represent new independent solutions, but can be expressed by both the functions $g_m^\pm$ through (22). In this case $\hat{g}_m$ would belong to $T_{m+1} \oplus T_{m-1}$.

Indeed, let $g_m^\pm$ be bases in $T_{m\pm1}$ and suppose that for some $\gamma_m = \hat{\gamma}_m$ there exists a function $\hat{g}_m$ which satisfies (26) but does not belong to $T_{m+1} \oplus T_{m-1}$.

We define a new function ($m \neq 0$),

$$u_m = \hat{g}_m - (2m)^{-1} \int_{S_{m+1}} \hat{\gamma}_m g_m^+ \, ds. \tag{28}$$

By (27) and (26) it satisfies equation (21), i.e.

$$L_{m-1} u_m = L_{m-1} \hat{g}_m - \int \hat{\gamma}_m g_m^+ \, ds = 0. \tag{29}$$

But, since the functions g_m^- constitute a base in T_{m-1}, u_m can be expressed as

$$u_m = \int_{S_{m-1}} \nu_m g_m^- \, ds. \tag{30}$$

Thus, taking into account (28), $\hat{g}_m$ must have a form

$$\hat{g}_m = \int_{S_{m-1}} \nu_m g_m^- \, ds + (2m)^{-1} \int_{S_{m+1}} \hat{\gamma}_m g_m^+ \, ds, \tag{31}$$

in contrast to our previous assumption. Therefore any solution of the fourth-order equation (17) can be written in the form

$$f_m = \int \alpha_m^+(s) g_m^+(s) \, ds + \int \alpha_m^-(s) g_m^-(s) \, ds, \tag{32}$$

and the solution of (16) as

$$F_\phi^m = f_m \, e^{im\phi}. \tag{33}$$

Obviously, this conclusion is true if $m \neq 0$.

The case $m = 0$ requires a separate discussion. It is shown in the appendix that one can interpret the functions $g_0^-(s)$ in (32) and (33) as $(dg_m^+/dm)_{m=0}$. However, if we intend to treat F_ϕ as a component of a physical field (electric or magnetic) in a source-free region, these last functions must be rejected.

So far, we have left essentially untouched the problem of how to find effectively the solutions of (21).

However, observe that if a scalar function

$$W_q = w_q(\xi_1, \xi_2) \, e^{iq\phi}, \tag{34}$$

satisfies the SH equation, w_q is the solution of the equation

$$L_q w_q = 0, \tag{35}$$

where the operator L_q is defined by (18). Comparing (35) with (21) we recognize that our desired functions $g_m^\pm$ are *de facto* the solutions of (35) but with shifted indices.

We find thus

$$g_m^\pm(\xi_1, \xi_2) = w_{m\pm1}(\xi_1, \xi_2), \tag{36}$$

where w_q satisfy the SH equation.

This is an effective technique for finding the solutions of (17) and (21), since the whole theory of the SH equation may be applied immediately.

The elementary solutions of (16) are then

$$F_{\phi}^{m}=\left(\int \alpha_{m}^{+} w_{m+1}\,\mathrm{d}s+\int \alpha_{m}^{-} w_{m-1}\,\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}m\phi}, \tag{37}$$

where w_q are defined as above, and the complete solution of (16) is the sum (1).

Obviously the boundary conditions, if imposed, determine the coefficients α_m^+ and α_m^- as well as the spectrum over which the integration in (37) is performed.

In § 4 we will apply the formal procedure presented above to physical fields in a source-free region. Observe however, that the procedure can be applied also to potentials, but in this case the functions g_0^- cannot be neglected.

4. The Maxwell equations in rotational coordinates

We will consider now the solutions of the Maxwell equations

$$\operatorname{curl}\boldsymbol{E}=\mathrm{i}k\boldsymbol{H}, \qquad \operatorname{curl}\boldsymbol{H}=-\mathrm{i}k\boldsymbol{E} \tag{38}$$

in the coordinates of class R. If the azimuthal field components E_3, H_3 are given by (1), we can easily express all other field components in terms of E_3 and H_3:

$$\begin{aligned}
E_1^m&=\mathrm{i}(M_m h)^{-1}[kh_3(h_3H_3^m)_{,2}+m(h_3E_3^m)_{,1}],\\
E_2^m&=\mathrm{i}(M_m h)^{-1}[m(h_3E_3^m)_{,2}+kh_3(h_3H_3^m)_{,1}],\\
H_1^m&=\mathrm{i}(M_m h)^{-1}[-kh_3(h_3E_3^m)_{,2}+m(h_3H_3^m)_{,1}],\\
H_2^m&=\mathrm{i}(M_m h)^{-1}[m(h_3H_3^m)_{,2}+kh_3(h_3E_3^m)_{,1}],
\end{aligned} \tag{39}$$

where $M_m=k^2h_3^2-m^2$, h is given by (6), and $E_{i,j}^m=\partial E_i^m/\partial\xi_j$.

Relations (39) are often applied in the literature, but confined rather to some particular cases (e.g. Fock and Fedorov 1958, Ivanov 1968).

By applying theorem (i) it is possible to construct functions E_ϕ and H_ϕ which are azimuthal components of vector fields satisfying the VH equation. E_ϕ will be given by (1), (37) and similarly

$$H_{\phi}^{m}=\left(\int \beta_{m}^{+}(s)g_{m}^{+}(s)\,\mathrm{d}s+\int \beta_{m}^{-}(s)g_{m}^{-}(s)\,\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}m\phi}. \tag{40}$$

Let us remark that the fields $\boldsymbol{E}$ having a component E_ϕ which satisfies (16) constitute a wider class than the class of the solutions of the VH equation. The same conclusion applies also to the fields $\boldsymbol{E}$, $\boldsymbol{H}$ determined by E_ϕ, and H_ϕ, through (39) and to the solutions of the Maxwell equations. Even if other field components are determined by (39), we must remember that there does not exist *a priori* any relationship between E_ϕ and H_ϕ, since both these functions are set independently as the solutions of (17).

For establishing the required relationship and/or for choosing the appropriate solutions we must go back either to the Maxwell equations, or to the divergence equations. In this manner we will be able to determine the necessary relations between the α and β in (37) and (40).

Substituting (39) into $\text{curl}_\phi \boldsymbol{E} = ikH_\phi$ we get

$$\{M_m^{-1}[m(h_3E_3^m)_{,2} - kh_3(h_3H_3^m)_{,1}]\}_{,1} - \{M_m^{-1}[m(h_3E_3^m)_{,1} + kh_3(h_3H_3^m)_{,2}]\}_{,2} = kh^2H_3^m. \tag{41}$$

Taking into account (7), after some re-arrangement we obtain the final formulae

$$M_m h_3^{-1} Q(H_3^m) - m^2(h_3^{-1}H_3^m + \text{grad}\, H_3^m \cdot \text{grad}\, h_3) + mkh_3h^{-2}(h_{3,1}E_{3,2}^m - h_{3,2}E_{3,1}^m) = 0, \tag{42}$$

and (from $\text{curl}_\phi \boldsymbol{H} = -ikE_\phi$)

$$M_m h_3^{-1} Q(E_3^m) - m^2(h_3^{-1}E_3^m + \text{grad}\, E_3^m \cdot \text{grad}\, h_3) - mkh_3h^{-2}(h_{3,1}H_{3,2}^m - h_{3,2}H_{3,1}) = 0 \tag{43}$$

where h is given by (6), M_m by (39), and the operator Q by (14). Equations (42) and (43) remove uncertainty in the determination of the azimuthal components E_ϕ, H_ϕ and thus equations (39) can be used to obtain the complete solutions of the Maxwell equations in rotational coordinates.

Whenever an electromagnetic field is derived from the two scalar functions, which represent the components of the electric and magnetic fields in the same direction, the problem of representation of the field in terms of TE and TM waves is automatically raised. In our case, TE or TM fields would be defined with respect to the ϕ direction (angularly transversal or azimuthally transversal).

From (42) and (43) it can be seen that such a decomposition is impossible, unless $m = 0$. All coefficients α and β can vanish only simultaneously and consequently an electromagnetic field for $m \neq 0$ has all the six components (curvilinear).

However, if $m = 0$, equations (42) and (43) are satisfied by $H^0 = \beta_0^+(s)g_0^+(s)$ and $E^0 = \alpha_0^+(s)g_0^+(s)$, respectively, ($g_0^-$ are excluded—see appendix). This result follows from the identity:

$$Q(f\, e^{\pm im\phi}) = e^{\pm im\phi}(L_{m+1} + m)f, \tag{44}$$

for arbitrary $f = f(\xi_1, \xi_2)$.

Thus, if $m = 0$, an electromagnetic field can be derived only from one azimuthal component, either E_ϕ or H_ϕ, and consequently the fields of TM or TE type with respect to the ϕ direction can be set.

To complete our discussion, observe that having a well defined azimuthal component E_ϕ one can easily find E_r as well as the Cartesian components E_x, E_y; this can be useful in some applications.

5. Conclusions

We have shown that in any rotational coordinate system belonging to class R because of the properties of the scale factors, the azimuthal component E_ϕ of any solution of the vector Helmholtz equation satisfies a partial differential equation of the fourth order.

If this component depends on the ϕ coordinate as $e^{\pm im\phi}$, the fourth-order equation can be factorized and, with one exception, its solutions can be expressed as linear combinations of the solutions of the factor equations.

Both second-order factor equations are of the same kind and are closely related to the scalar Helmholtz equation.

Because of the important role played by the azimuthal coordinate, it is possible to derive all components of the electromagnetic field from the two scalar functions which are the solutions of the fourth-order equation already mentioned.

If these solutions are properly chosen, they can be identified with the azimuthal components E_ϕ, and H_ϕ of the electromagnetic field and the calculation of the other components (curvilinear) is guaranteed by suitable formulae.

Electromagnetic fields will possess as a rule all six curvilinear components. However, the TE and TM fields, with respect to the ϕ direction, can also exist.

Thus, we have shown that besides the z components in cylindrical coordinate systems, radial components in spherical coordinate systems, Cartesian components in rectangular coordinates and also the azimuthal components in any rotational coordinate system of class R permit the determination of effectively the whole electromagnetic field from just two scalar functions; however, in general, these functions are not independent.

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Appendix

Let $g_0^+(s)$, $s \in S_0$, represent a base in a space of the solutions of the equation

$$L_1(g_0^+) = 0; \tag{A.1}$$

this equation can be treated as a limiting case of

$$\lim_{m \to 0} L_{1+m}(g_m^+) = 0. \tag{A.2}$$

Forgetting subtleties, by differentiation of (A.2) with respect to m, and from (18), for $m = 0$, we have

$$L_1(g_0) = -g_0^+, \tag{A.3}$$

where

$$g_0 = \left(\frac{\mathrm{d}}{\mathrm{d}m} g_m^+\right)_{m=0}. \tag{A.4}$$

Observe that (A.3) is identical with (24). By (A.3), $g_0(s)$ is independent of $g_0^+(s)$ and evidently it satisfies the equation

$$L_1^2(g_0) = 0. \tag{A.5}$$

Since for $m = 0$, both equations (21) reduce to (A.1), we can just treat functions g_0 as a second independent solution and denote eventually by g_0^-.

It is not difficult to show that g_0 cannot represent an azimuthal component of an electromagnetic field but can only represent potential fields ($k=0$). To prove this statement, note that by (44) and (A.3), the first term in (42) is

$$Mh_3^{-1}Q(g_0)=k^2h_3Q(g_0)=k^2h_3L_1(g_0)=-k^2h_3g_0^+\neq 0, \qquad \text{(A.6)}$$

if $k\neq 0$, whereas all other terms in (42) vanish, since g_0 does not depend on ϕ.

Consequently, (42) and (43) are not satisfied and g_0 must be excluded from the set of functions which represent the azimuthal components of a dynamical electromagnetic field in a source-free region.

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